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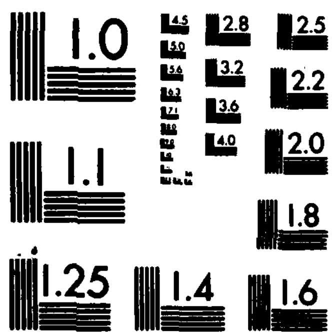

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THE APPROXIMATION THEORY FOR THE P-VERSION OF THE FINITE ELEMENT METHOD, II

by

Milo R. Dorr

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This report extends the results of Technical Note BN-1004. It analyzes the performance of the p-version of the finite element method for 3-dimensional elasticity problems in domains with edges and corners.

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THE APPROXIMATION THEORY FOR THE P-VERSION OF THE FINITE ELEMENT METHOD, II

by

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Abstract

In Part II of this paper, the approximation theory developed in Part I 9~~[11]~~ is used to determine the piecewise polynomial approximability of solutions of elliptic problems on polygonal domains in R^2 and polyhedra in R^3. From these estimates, convergence orders for the p-version of the finite element method applied to such problems are readily obtained. The critical issue is the approximation of the singularities which occur at the non-smooth parts of the domain boundaries. It is seen that the estimates of ~~[11] involving the weighted Sobolev spaces~~ Z^s_ℓ are well-suited for treating such singular functions, yielding directly the optimal approximation degree, up to an arbitrarily small ε.

Numerical results for two problems from two-dimensional linear elasticity are also presented. The computations show that the predicted order of convergence is achieved even for low values of p. Moreover, in contrast to the usual h-version of the finite element method, the point at which the p-version enters the asymptotic range does not depend on problem parameters such as the Poisson ratio.

1. Introduction

In this paper, the approximation theory developed in [11] for the p-version of the finite element method is applied to elliptic problems in polygonal and polyhedral domains. Assuming that the p-version (see e.g. [6]) is used to compute an approximate piecewise polynominal solution to some elliptic problem, the question at issue is the rate at which the approximate solution will converge to the true solution in the appropriate energy space as the polynomial degree p is increased. It is well-known that if a coercivity or "inf-sup" condition [2,10] exists for the given elliptic operator, then this question reduces to a purely approximation-theoretic problem of the kind treated in [11]. The purpose of the present paper is to show how the results of [11] may be used to determine the approximability of solutions of some model problems in the usual Sobolev spaces by piecewise polynomials satisfying appropriate boundary and conformality conditions.

As with any approximation result, the degree of approximation for the p-version is determined by the regularity of the function being approximated. The regularity of solutions of elliptic problems on smooth domains is classical [1,19], unfortunately few problems of practical interest fall into this category. For elliptic problems on non-smooth domains such as polygons and polyhedra, one must turn to the regularity theory developed e.g. in [12-18]. Essentially, these results show that solutions on such domains may be decomposed into the sum of smooth functions and functions which possess singular derivatives at the corners (and edges in R^3) of the domain. It suffices therefore to consider the approximability of these singular functions, whose form can usually be given very explicitly. It has been previously noted [6] that, unlike the h-version estimates, the approximation results for such singular functions as obtained from estimates involving the usual unweighted Sobolev spaces H^s are not optimal for the p-version. As will be seen in the following sections, however, the singular character of these functions is well-distinguished by the weighted Sobolev spaces Z^s_ℓ introduced in [11], and optimal results are obtained by the estimates of [11]. This is clearly due to the close relationship between the weighted spaces Z^s_ℓ and polynomial approximation indicated in [11].

In Section 2 the notation and main approximation results of [11] are reviewed. Section 3 addresses the approximation of the solution of a model elliptic boundary-value problem in a polygon. This two-dimensional result has been previously obtained in [6] by a different method not involving weighted spaces. The advantage of the current approach lies in the fact that more general kinds of singularities can be analyzed using the weighted spaces Z^s_ℓ than with the techniques of [6]. In particular, one may allow a continuum of singularities along part of the boundary of the domain. This situation occurs in three-dimensional problems along edges of polyhedral domains as described in Section 4, where some new approximation results are obtained. In Section 5, some numerical results are presented for a sample of problems in two-dimensional linear elasticity. These were computed by the research program COMET-X [7] developed by the Center for Computational Mechanics at Washington University in St. Louis.

2. Review of the Approximation Results

The following is a summary of the notation and main results of [11].

For each positive integer n, let $I^n = \{x = (x_1, \ldots, x_n): -1 < x_i < 1, 1 \le i \le n\}$. For each non-negative real number s such that $s \neq \frac{1}{2}$ + an integer, define

$$Z^s(I^n) = \{u: \|u\|_{Z^s(I^n)} < \infty\}$$

where, if s = k an integer, then

$$\|u\|_{Z^s(I^n)} = \left(\int_{I^n} |u|^2 \, dx + \sum_{i=1}^{n} \int_{I^n} \left| \frac{\partial^k u}{\partial x_i^k} \right|^2 (1 - x_i^2)^k \, dx \right)^{1/2} ,$$

and if $s = k + \beta$ with k an integer and $0 < \beta < 1$ ($\beta \neq 1/2$), then

$$\|u\|_{Z^s(I^n)} = \Bigg(\|u\|^2_{Z^k(I^n)}$$

$$+ \sum_{i=1}^{n} \int_{I^{n-1}} \int_{I \times I} \Big| (1 - t^2)^{s/2} \frac{\partial^k u}{\partial x_i^k} (x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$$

$$- (1 - \tau^2)^{s/2} \frac{\partial^k u}{\partial x_i^k} (x_1, \ldots, x_{i-1}, \tau, x_{i+1}, \ldots, x_n) \Big|^2$$

$$\cdot |t - \tau|^{-1-2\beta} \, dt \, d\tau \, dx_1 \ldots dx_{i-1} \, dx_{i+1} \ldots dx_n \Bigg)^{1/2}.$$

Let Ω be a domain in R^n such that there exists a triangulation Δ of Ω into open n-simplices Ω_i, $i=1, \ldots, M$. Let σ^ν, $\nu = 1, \ldots, N$, denote the vertices of Δ. Since it is convenient to be able to refer to the vertices of a particular simplex Ω_i, for $i=1, \ldots, M$ let $\sigma_{i,j}$, $1 \leq j \leq n + 1$, denote the vertices of Ω_i. Hence, if a vertex σ^ν of Δ is also a vertex of Ω_i, then $\sigma^\nu = \sigma_{i,j}$ for some j. Let $S_\Delta = \{(i,j): 1 \leq i \leq M, 1 \leq j \leq n + 1\}$.

Consider a simplex $\Omega_i \in \Delta$ and one of its vertices $\sigma_{i,j}$. Letting $e_1, \ldots, e_n$ be the vectors based at $\sigma_{i,j}$ and terminating at the other n vertices of Ω_i, define in R^n the parallelepiped

$$\Omega_{i,j} = \{ x \in R^n : \; x = \sigma_{i,j} + \sum_{m=1}^{n} t_m e_m, \; 0 < t_m < 1 \} .$$

Clearly, $\Omega_i \subset \Omega_{i,j}$ for all $j=1, \ldots, n+1$. For each $(i,j) \in S_\Delta$ choose an affine mapping $T_{i,j}$ in R^n which maps $\Omega_{i,j}$ onto I^n and $\sigma_{i,j}$ onto the point $(1, \ldots, 1)$.

Let $\{\eta_\nu\}_{\nu=1}^N$ be a smooth partition of unity on $\bar\Omega$ such that for each $\nu=1, \ldots, N$, supp η_ν contains the vertex σ^ν and supp η_ν intersects only those closed simplices $\bar\Omega_i$ which have σ_ν as a vertex. For any $u \in L_2(\Omega)$ and $(i,j) \in S_\Delta$, define in $\Omega_{i,j}$

$$u_{i,j} = \begin{cases} u\eta_\nu & \text{in } \bar\Omega_i \text{ where } \nu \text{ is such that } \sigma_{i,j} = \sigma^\nu, \\ 0 & \text{in } \overline{\Omega_{i,j}/\Omega_i} \ . \end{cases}$$

By the assumptions on η_ν, one observes that $u_{i,j} = u$ in a neighborhood of $\sigma_{i,j}$ and $u_{i,j} \equiv 0$ outside of a neighborhood G of $\sigma_{i,j}$ such that $G \cap \overline{\Omega_{i,j}/\Omega_i} = \phi$.

For each real $s \geq 0$ such that $s \neq \frac{1}{2}$ + an integer, let

$$Z^s(\Omega;\Delta) = \{u : \|u\|_{Z^s(\Omega;\Delta)} < \infty\}$$

where

$$\|u\|_{Z^s(\Omega;\Delta)} = \left(\sum_{(i,j)\in S_\Delta} \|u_{i,j} \circ T_{i,j}^{-1}\|^2_{Z^s(I^n)}\right)^{1/2} .$$

For each non-negative integer p, let $P_p(\Omega_i)$, $1 \leq i \leq M$, denote the space of all polynomials on Ω_i of degree at most p, and let

$$P_p(\Omega;\Delta) = \{u : u|_{\Omega_i} \in P_p(\Omega_i),\ 1 \leq i \leq M\} \quad .$$

For all non-negative integers ℓ and p, set

$$Z^s_\ell(\Omega;\Delta) = Z^s(\Omega;\Delta) \cap C^\ell(\bar\Omega) \quad ,$$

$$P^\ell_p(\Omega;\Delta) = P_p(\Omega;\Delta) \cap C^\ell(\bar\Omega) \quad ,$$

where $C^\ell(\bar\Omega)$ denotes the set of all functions which along with their first ℓ derivatives are continuous on $\bar\Omega$. $P^\ell_p(\Omega;\Delta)$ is then the set of all functions in $P_p(\Omega;\Delta)$ which along with their first ℓ derivatives are continuous across the common boundaries of adjacent simplices of Δ. It is shown in [11] that if $0 \leq \ell < \frac{s-n}{2}$ then the analogous statement holds for $Z^s_\ell(\Omega;\Delta)$ and $Z^s(\Omega;\Delta)$.

Recalling that Ω is the union of n-simplices Ω_i, let Γ_k, k=1, ..., K, denote the faces of Ω, i.e., the (n-1)-simplices whose union is the boundary of Ω. Fix $s \geq 0$ and let ℓ^* be the largest integer strictly less than $\frac{s-n}{2}$. For each k=1, ..., K, let D_k be a subset of $\{0, 1, \ldots, \ell^*\}$. Denoting by D the family $\{D_k;\ 1 \leq k \leq K\}$, define

$$Z^s_\ell(\Omega;\Delta,D) = \{u \in Z^s_\ell(\Omega;\Delta) : \text{ for } k=1, \ldots, K,$$

$$\frac{\partial^m u}{\partial n^m} = 0 \text{ on } \Gamma_k \text{ for each } m \in D_k \ \}$$

and

$$P^\ell_p(\Omega;\Delta,D) = \{u \in P^\ell_p(\Omega;\Delta) : \text{ for } k=1, \ldots, K,$$

$$\frac{\partial^m u}{\partial n^m} = 0 \text{ on } \Gamma_k \text{ for each } m \in D_k \ \}$$

where $\frac{\partial^0 u}{\partial n^0} = u$ and, for $m \geq 1$, $\frac{\partial^m u}{\partial n^m}$ is the m^{th} order derivative of u with respect to n, the outward pointing unit vector normal to the boundary of Ω.

Letting $H^s(\Omega)$ be the usual Sobolev space of order s on Ω and setting $Z^s_{-1}(\Omega;\Delta,D) = Z^s(\Omega;\Delta)$, the main result of [11] may now be stated:

<u>Theorem 2.1</u> Let s and s' be such that $s > 2s' \geq 0$ and s, $2s' \neq \frac{1}{2}$ + an integer. Let ℓ^* denote the largest integer strictly less than $\frac{s-n}{2}$. If $u \in Z^s_\ell(\Omega;\Delta,D)$ for some integer ℓ satisfying $s' - \frac{3}{2} < \ell < \ell^*$ and for some D, then for each non-negative integer p there exists $\varphi_p \in P^\ell_p(\Omega;\Delta,D)$ such that for arbitrarily small $\varepsilon > 0$,

$$(2.1) \qquad \|u - \varphi_p\|_{H^{s'}(\Omega)} \leq Cp^{-s+2s'+\varepsilon}\|u\|_{Z^s(\Omega;\Delta)}$$

where $C = C(s, s', \varepsilon)$ is independent of u and p. Moreover, if $u \in Z^s_{\ell *}(\Omega; \Delta, D)$ then for any integer $\ell > s' - 3/2$ and for any non-negative integer p there exists $\varphi_p \in P^{\ell}_p(\Omega; \Delta, D)$ such that (2.1) holds.

For any real $s \geq 0$, define $D = \{D_k\}$ as above and let

$$H^s(\Omega;D) = \{u \in H^s(\Omega) : \text{ for } k=1, \ldots, K,$$

$$\frac{\partial^m u}{\partial n^m} = 0 \text{ on } \Gamma_k \text{ for each } m \in D_k\} \quad .$$

The following [11, Theorem 3.4] is a consequence of Theorem 2.1:

<u>Theorem 2.2</u> Let s and s' be real numbers such that $s > s' \geq 0$. If $u \in H^s(\Omega;D)$, then for each non-negative integer p and each integer $\ell > s' - 3/2$ there exists $\varphi_p \in P^{\ell}_p(\Omega; \Delta, D)$ such that for arbitrarily small $\varepsilon > 0$,

$$\|u - \varphi_p\|_{H^{s'}(\Omega)} \leq Cp^{-s+s'+\varepsilon} \|u\|_{H^s(\Omega)}$$

where $C = C(s,s',\varepsilon)$ is independent of u and p.

<u>Remarks</u>:

1. Although boundary conditions were not explicitly included in [11, Theorems 3.3 and 3.4], their proofs are easily modified as indicated in [11] to obtain Theorems 2.1 and 2.2.

2. Theorem 2.2 was previously obtained in [6] for the case $s'=1$ and $H^s(\Omega;D) = H^s(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$, $s > 1$.

3. The Approximation of Solutions of Elliptic Boundary-Value Problems in Polygonal Domains

Let Ω denote a polygonal domain in R^2 with boundary Γ. Then Γ is the union of a finite number of line segments Γ_ℓ, $1 \le \ell \le L$, which one may assume to be labelled in consecutive order following the positive orientation of Γ. Assume also that the set $\{1, 2, \ldots, L\}$ is partitioned into two subsets D and N, and let $\Gamma_D = \bigcup_{\ell \in D} \Gamma_\ell$ and $\Gamma_N = \bigcup_{\ell \in N} \Gamma_\ell$. For each ℓ, $1 \le \ell \le L$, let n_ℓ denote the outward pointing unit normal to Γ_ℓ, let $\beta_\ell = \frac{\pi}{2}$ if $\ell \in D$, and let $\beta_\ell = 0$ if $\ell \in N$.

Consider the following model problem. Given k a non-negative integer, $f \in H^k(\Omega)$, and $g_\ell \in H^{k+1/2}(\Gamma_\ell)$ if $\ell \in N$, find u such that

$$(3.1)\qquad \begin{aligned} -\Delta u + u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_D \\ \frac{\partial u}{\partial n_\ell} &= g_\ell && \text{on } \Gamma_\ell, \quad \ell \in N \quad . \end{aligned}$$

Letting $H^1(\Omega;D) = \{u \in H^1(\Omega): u = 0 \text{ on } \Gamma_D\}$, (3.1) is equivalent (see e.g. [13]) to the following variational problem: Find $u \in H^1(\Omega;D)$ such that

$$(3.2)\qquad \int_\Omega (\nabla u \cdot \nabla v + uv)dx - \sum_{\ell \in N} \int_{\Gamma_\ell} g_\ell v \, d\Gamma = \int_\Omega f v \, dx$$

$$\text{for all } v \in H^1(\Omega;D) \quad .$$

It is known [13] that (3.2) admits a unique solution u.

The present objective is to determine the approximability of the solution u in $H^1(\Omega)$ by elements of $P_p^0(\Omega;\Delta,D)$ where Δ is some triangulation of Ω. In order to use the results of the previous section, one must know the regularity of u. Since $f \in H^k(\Omega)$, it might be expected that $u \in H^{k+2}(\Omega)$. However, it is well-established that this is not true, in general, due to possible singularities at the corners of Ω and at points of Γ where a change of boundary condition occurs. Nevertheless, under certain conditions, the solution u may be decomposed into the sum of a function in $H^{k+2}(\Omega)$ and functions which possess singular derivatives at the corners and points of changing boundary conditions. In order to give a precise statement of this result, for each ℓ, $1 \leq \ell \leq L$, let σ_ℓ denote the common endpoint of Γ_ℓ and $\Gamma_{\ell+1}$ (or Γ_L and Γ_1 if $\ell=L$), and let ω_ℓ, $0 < \omega_\ell < 2\pi$, denote the measure of the interior angle of Ω at σ_ℓ. It will be assumed that if $\omega_\ell = \pi$ then either $\ell \in D$, $\ell+1 \in N$ or else $\ell \in N$, $\ell+1 \in D$ (with the obvious modification for $\ell=L$); i.e., if $\omega_\ell = \pi$ then the boundary conditions in (3.1) change from Dirichlet to Neumann or vice-versa at σ_ℓ. For each ℓ, $1 \leq \ell \leq L$, let (r_ℓ, θ_ℓ) denote polar coordinates based at σ_ℓ such that $\theta_\ell=0$ contains the segment $\Gamma_{\ell+1}$. Finally, for each ℓ, $1 \leq \ell \leq L$, and each integer m, let

$$\lambda_{\ell,m} = \frac{\beta_{\ell+1} - \beta_\ell + m\pi}{\omega_\ell} .$$

The following is a consequence of [13]:

<u>Lemma 3.1</u> Let k be a non-negative integer, let $f \in H^k(\Omega)$, and let $g_\ell \in H^{k+1/2}(\Gamma_\ell)$ if $\ell \in N$. If $\dfrac{\beta_{\ell+1} - \beta_\ell - (k+1)\omega_\ell}{\pi}$ is not an integer for any ℓ, then there exists $w \in H^{k+2}(\Omega)$ and numbers $c_{\ell,m}$ such that

$$u = w + \sum_{\substack{0<\lambda_{\ell,m}<k+1 \\ \lambda_{\ell,m} \neq \text{ an integer}}} c_{\ell,m} r_\ell^{\lambda_{\ell,m}} \cos(\lambda_{\ell,m}\theta_\ell - \beta_{\ell+1}) \chi_\ell(r_\ell, \theta_\ell)$$

(3.3)

$$+ \sum_{\substack{0<\lambda_{\ell,m}<k+1 \\ \lambda_{\ell,m} = \text{an integer}}} c_{\ell,m} r_\ell^{\lambda_{\ell,m}} \{\ell n\, r_\ell \cos(\lambda_{\ell,m}\theta_\ell-\beta_{\ell+1})-\theta_\ell \sin(\lambda_{\ell,m}\theta_\ell-\beta_{\ell+1})\}\chi_\ell(r_\ell,\theta_\ell)$$

is the unique solution of (3.1), where the χ_ℓ are smooth functions equal to unity near σ_ℓ and zero away from σ_ℓ. Moreover,

$$(3.4) \qquad \|w\|_{H^{k+2}(\Omega)} + \sum |c_{\ell,m}| \leq C\, (\|f\|_{H^k(\Omega)} + \sum_{\ell\in N} \|g_\ell\|_{H^{k+1/2}(\Gamma_\ell)}) \ .$$

The importance of Lemma 3.1 lies in the fact that the piecewise polynomial approximability of u may now be determined by investigating the approximability of each term in the right-hand side of (3.3) separately (with a slight modification) using the results of Section 2. The following lemma is required:

<u>Lemma 3.2</u> Let ℓ be such that $1 \leq \ell \leq L$, let $\eta_\ell \in C^\infty(\bar{\Omega})$ be such that supp η_ℓ is contained in a neighborhood of σ_ℓ, and let ξ_1, $\xi_2 \in C^\infty([0, \omega_\ell])$. If μ is any non-negative real number and if

$$v(r_\ell,\theta_\ell) = r_\ell^\mu(\ell n\, r_\ell\, \xi_1(\theta_\ell) + \xi_2(\theta_\ell)) \quad ,$$

then $v\eta_\ell \in Z_0^{2\mu+2-\varepsilon}(\Omega;\Delta)$ for arbitrarily small $\varepsilon > 0$.

<u>Pf</u>: Let (r,θ) denote a polar coordinate system centered at the point (1,1); in particular, let $r = [(x_1-1)^2+(x_2-1)^2]^{1/2}$ and let θ be the angle measured from the semi-axis $\{x\colon\ x_1 < 1,\ x_2 = 1\}$. It suffices to show that if ξ_1, $\xi_2 \in C^\infty([0, \frac{\pi}{2}])$, then the function

$$v(r,\theta) = r^\mu(\ell n\, r\ \xi_1(\theta) + \xi_2(\theta)) \quad , \quad (r,\ \theta) \in I^2 \ ,$$

belongs to $Z^{2\mu+2-\varepsilon}(I^2)$ for arbitrarily small $\varepsilon > 0$.

To this end, let $\chi \in C^\infty([0,\infty))$ be such that

$$\chi(t) = \begin{cases} 0 & \text{for} \quad 0 \le t < 1/2 \\ 1 & \text{for} \quad 1 \le t < \infty \end{cases}$$

and for each $\delta > 0$ let $\chi_\delta(t) = \chi(\frac{t}{\delta})$. Setting $v_\delta = v\chi_\delta(r)$, it follows by Leibniz' rule that for each integer $k > 2\mu + 2$ and for $i = 1,2$,

$$\int_{I^2} \left| \frac{\partial^k v_\delta}{\partial x_i^k} \right|^2 (1 - x_i^2)^k \, dx \le C \sum_{m=0}^{k} \int_{I^2} \left| \frac{\partial^m v}{\partial x_i^m} \right|^2 \left| \frac{\partial^{k-m} \chi_\delta}{\partial x_i^{k-m}} \right|^2 (1 - x_i)^k \, dx$$

$$\le C \left(\sum_{m=0}^{k-1} \delta^{-2k+2m} \int_{\delta/2}^{\delta} r^{2\mu-2m+k+1-\varepsilon} \, dr + \int_{\delta/2}^{1} r^{2\mu-k+1-\varepsilon} dr \right)$$

$$\le C \, \delta^{2\mu-k+2-\varepsilon} .$$

Furthermore, since

$$\|v_\delta\|_{L_2(I^2)} \le C \, \|v\|_{L_2(I^2)} ,$$

one obtains that for any integer $k > 2\mu + 2$,

$$\|v_\delta\|_{Z^k(I^2)} \le C \, \delta^{\mu - \frac{k}{2} + 1 - \frac{\varepsilon}{2}}. \tag{3.6}$$

Now,

$$\int_{I^2} |v - v_\delta|^2 \, dx = \int_{I^2} |v(1 - \chi_\delta)|^2 \, dx$$

$$\le C \int_0^\delta r^{2\mu+1-\varepsilon} \, dr$$

$$\le C \, \delta^{2\mu+2-\varepsilon} ,$$

i.e.,

$$\|v - v_\delta\|_{L_2(I^2)} \le C \, \delta^{\mu+1-\frac{\varepsilon}{2}} . \tag{3.7}$$

For each $t > 0$, let

$$K(v,t) = \inf_{v=v_1+v_2} (\|v_1\|_{L_2(I^2)} + t\,\|v_2\|_{Z^k(I^2)}) \quad .$$

If $0 < t < 1$, then by choosing $\delta = t^{2/k}$ it follows from (3.6) and (3.7) that for $k > 2\mu + 2$,

$$K(v,t) \le \|v - v_\delta\|_{L_2(I^2)} + t\,\|v_\delta\|_{Z^k(I^2)}$$

$$\le C\, t^{\frac{2\mu+2-\varepsilon}{k}} \quad .$$

Furthermore, for $t \ge 1$,

$$K(v,t) \le \|v\|_{L_2(I^2)} \quad .$$

Hence, for arbitrarily small $\varepsilon > 0$,

$$\int_0^\infty (t^{-\frac{2\mu+2-\varepsilon}{k}} K(v,t))^2 \frac{dt}{t} \le C\int_0^1 t^{\frac{2\varepsilon}{k}-1} dt$$

$$+ \|v\|^2_{L_2(I^2)} \int_1^\infty t^{-\frac{2\mu+2-\varepsilon}{k}-1} dt < \infty \quad .$$

This implies (see e.g. [9]) that $v \in (L_2(I^2), Z^k(I^2))_{\frac{2\mu+2-\varepsilon}{k},\,2}$ and the result follows from [11, Theorem 2.1].

<u>Theorem 3.1</u> Let k be a non-negative integer, let $f \in H^k(\Omega)$, let $g_\ell \in H^{k+1/2}(\Gamma_\ell)$ if $\ell \in N$, and assume that $\frac{\beta_{\ell+1}-\beta_\ell-(k+1)\omega_\ell}{\pi}$ is not an integer for any ℓ. If u is the solution of (3.1) then for each non-negative integer p there exists $\psi_p \in P^0_p(\Omega;\Delta,D)$ such that

$$\text{(3.8)} \qquad \|u - \psi_p\|_{H^1(\Omega)} \le Cp^{-\min\{2\lambda_{1,1},\ \ldots,\ 2\lambda_{L,1},k+1\} + \varepsilon} (\|f\|_{H^k(\Omega)}$$

$$+ \sum_{\ell \in N} \|g_\ell\|_{H^{k+1/2}(\Gamma_\ell)})$$

for arbitrarily small $\varepsilon > 0$, where $C = C(\varepsilon,k)$ is independent of u and p.

<u>Pf</u>: Let $\lambda_{\ell,m}$, $c_{\ell,m}$, w, and χ_ℓ be as in Lemma 3.1. For each ℓ and m such that $0 < \lambda_{\ell,m} < k+1$ and $\lambda_{\ell,m} \neq$ an integer, let

$$z_{\ell,m}(r_\ell,\theta_\ell) = r_\ell^{\lambda_{\ell,m}} \cos(\lambda_{\ell,m}\theta_\ell - \beta_{\ell+1})\ \chi_\ell(r_\ell,\theta_\ell) \quad .$$

For each ℓ and m such that $0 < \lambda_{\ell,m} < k+1$ and $\lambda_{\ell,m}$ = an integer, define

$$\xi_{\ell,m}(\theta_\ell) = \begin{cases} (-1)^{m+1}\ \omega_\ell \left(\dfrac{\sin\theta_\ell}{\sin\omega_\ell}\right)^{\lambda_{\ell,m}} & \text{if } \omega_\ell \neq \pi \\[2ex] (-1)^{m+1}\ \pi\ (\sin\theta_\ell)^{\lambda_{\ell,m}} & \text{if } \omega_\ell = \pi,\ \ell+1 \in N, \text{ and } \ell \in D \\[2ex] (-1)^{m+1}\ \pi\ (-\cos\theta_\ell)^{\lambda_{\ell,m}} & \text{if } \omega_\ell = \pi,\ \ell \in D, \text{ and } \ell+1 \in N \end{cases}$$

and set

$$z_{\ell,m}(r_\ell,\theta_\ell) = r_\ell^{\lambda_{\ell,m}} \{\ln r_\ell \cos(\lambda_{\ell,m}\theta_\ell - \beta_{\ell+1}) - \theta_\ell \sin(\lambda_{\ell,m}\theta_\ell - \beta_{\ell+1}) + \xi_{\ell,m}(\theta_\ell)\}\ \chi_\ell(r_\ell,\theta_\ell) \ .$$

Finally, let

$$\tilde{w} = w - \sum_{\substack{0<\lambda_{\ell,m}<k+1 \\ \lambda_{\ell,m} = \text{ an integer}}} c_{\ell,m}\, r_\ell^{\lambda_{\ell,m}}\, \xi_{\ell,m}\, \chi_\ell \quad .$$

Then

$$u = \tilde{w} + \sum_{0<\lambda_{\ell,m}<k+1} c_{\ell,m}\, z_{\ell,m} \quad . \tag{3.9}$$

Moreover $\tilde{w} \in H^{k+2}(\Omega) \cap H^1(\Omega;D)$ and by Lemma 3.2, $z_{\ell,m} \in Z_0^{2\lambda_{\ell,m}+2-\varepsilon}(\Omega;\Delta,D)$ for arbitrarily small $\varepsilon > 0$. Applying Theorem 2.2 to $\tilde{w}$ and Theorem 2.1 to the $z_{\ell,m}$, it follows that for each non-negative integer p there exists $\tilde{\psi}_p \in P_p^0(\Omega;\Delta,D)$ and $\psi_p^{(\ell,m)} \in P_p^0(\Omega;\Delta,D)$, $0 < \lambda_{\ell,m} < k+1$, such that

$$(3.10)\qquad \|\tilde{w} - \tilde{\psi}_p\|_{H^1(\Omega)} \le Cp^{-k-1+\varepsilon} \|\tilde{w}\|_{H^{k+2}(\Omega)}, \text{ and}$$

$$(3.11)\qquad \|z_{\ell,m} - \psi_p^{(\ell,m)}\|_{H^1(\Omega)} \le Cp^{-2\lambda_{\ell,m}+\varepsilon} \|z_{\ell,m}\|_{Z^{2\lambda_{\ell,m}+2-\varepsilon}(\Omega;\Delta)},\ 0 < \lambda_{\ell,m} < k+1,$$

where C is independent of p. Setting

$$(3.12)\qquad \psi_p = \tilde{\psi}_p + \sum_{0<\lambda_{\ell,m}<k+1} c_{\ell,m}\, \psi_p^{(\ell,m)} ,$$

(3.8) follows from (3.9) through (3.12) together with (3.4) and the triangle inequality.

Remarks 1. Theorem 3.1 for the case $\Gamma_D = \Gamma$, $\Gamma_N = \emptyset$ is proven in [6, Theorem 4.3]. The techniques employed there, however, do not use weighted spaces to obtain the result; instead, the singular functions are approximated directly.

2. Theorem 3.1 shows the doubled order of convergence obtained by using the p-version of the finite element method instead of the h-version with quasi-uniform refinement. Letting N denote the dimension of $P_p^0(\Omega;\Delta,D)$ and letting h denote the maximum diameter of any triangle in Δ, it holds that

$$(3.13)\qquad N \sim \begin{cases} h^{-2} & \text{for the h-version (quasi-uniform refinement),} \\ p^2 & \text{for the p-version.} \end{cases}$$

It can be shown (see e.g. [4]) that functions of the form (3.5) belong to the space $H^s(\Omega)$ iff $1 \le s < \lambda + 1$. Letting

$$\mu = \min_{0<\lambda_{\ell,m}<k+1} \lambda_{\ell,m}$$

and assuming that k is large (which it usually is in practice), it follows from the standard h-version approximation theory [2,10] that the approximation error in $H^1(\Omega)$ for the h-version is of order $h^{\mu} \sim N^{-\mu/2}$. As for the p-version, Theorem 3.1 together with (3.13) yield an approximation error of order $p^{-2\mu+\varepsilon} \sim N^{-\mu+\varepsilon}$ for arbitrarily small $\varepsilon > 0$.

Results such as Theorem 3.1 may similarly be obtained for elliptic operators other than that of (3.1) provided that results such as Lemma 3.1 are available for such operators. Consider for example the biharmonic problem: given $f \in H^k(\Omega)$, $k \geq 0$, find u such that

$$
(3.14) \qquad \begin{aligned} \Delta^2 u &= f && \text{in } \Omega \\ u = \partial u/\partial n &= 0 && \text{on } \Gamma. \end{aligned}
$$

Letting $H^2(\Omega;D) = \{ u \in H^2(\Omega) : u = \partial u/\partial n = 0 \text{ on } \Gamma\}$, one seeks to approximate u in $H^2(\Omega)$ by elements of $P_p^1(\Omega;\Delta,D) = H^2(\Omega;D) \cap P_p(\Omega;\Delta)$. In [18] it is shown that the solution u of (3.14) may be written as

$$
u = w + \sum_{0 < \mathrm{Re}\, \lambda_{\ell,m} < \alpha(k)} r^{\lambda_{\ell,m}} \; Q_{\ell,m}(r_\ell)\chi_\ell(r_\ell,\theta_\ell)
$$

where

(i) $w \in H^{k+4}(\Omega;D)$,

(ii) the $\lambda_{\ell,m}$ are the solutions of the nonlinear equation

$$
\sin^2(\lambda_{\ell,m} - 1)\omega_\ell - (\lambda_{\ell,m} - 1)^2 \sin^2\omega_\ell = 0,
$$

(iii) the functions $Q_{\ell,m}$ are of the form

$$Q_{\ell,m}(r_\ell) = \sum_j c_{m,j}(\theta_\ell)\, r_\ell^{\eta_{m,j}}\, \ln^{\beta_{m,j}} r_\ell$$

for some smooth functions $c_{m,j}$, non-negative integers $\eta_{m,j}$, and real numbers $\beta_{m,j} \geq 0$, and

(iv) χ_ℓ is a smooth cutoff function equal to unity near σ_ℓ and vanishing away from σ_ℓ.

Then the following may be obtained in a manner similar to Theorem 3.1:

Theorem 3.2 Let $f \in H^k(\Omega)$ for some non-negative integer k and let u be the solution of (3.14). Then for each non-negative integer p there exists $\psi_p \in P_p^1(\Omega;\Delta,D)$ such that

$$\| u - \psi_p \|_{H^2(\Omega)} \leq Cp^{-\min\{2 \min_{\ell,m} \mathrm{Re}\lambda_{\ell,m} - 2,\, k+2\} + \epsilon} \|f\|_{H^k(\Omega)}$$

for arbitrarily small $\epsilon > 0$, where $C = C(\epsilon,k)$ is independent of u and p.

4. The Approximation of Solutions of Elliptic Boundary-Value Problems in Polyhedral Domains

Let Ω denote a polyhedral domain in R^3 with boundary Γ and consider the following model problem: Given f in $C^\infty(\bar{\Omega})$, find u such that

$$(4.1)\qquad \begin{aligned} -\Delta u + u &= f \qquad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \Gamma . \end{aligned}$$

Letting

$$H^1(\Omega;D) = \{u \in H^1(\Omega): \ u = 0 \text{ on } \Gamma\} \quad ,$$

it is well-known that (4.1) admits a unique variational solution $u \in H^1(\Omega;D)$. In order to determine the approximability of u in $H^1(\Omega)$ by elements of $P_p^0(\Omega;\Delta,D)$ for some triangulation Δ of Ω, the regularity of u must again be established. By classical elliptic regularity theory [1,19], the solution u of (4.1) is infinitely differentiable in the interior of Ω and up to the smooth faces of Γ. However, in analogy with the previous two-dimensional case, u will, in general, possess singular derivatives at the corners and edges of Ω.

Consider first the behavior of u near an edge of Ω. Let γ denote an open segment such that $\bar{\gamma}$ is contained in the interior of some edge of Ω. Letting $x = (x_1, x_2, x_3)$ denote Cartesian coordinates in R^3, assume without loss of generality that γ is contained in the positive x_3-axis. Let ω denote the measure of the interior dihedral angle formed by the two faces Γ_1 and Γ_2 of Ω which intersect along γ. Let (r, φ, x_3) denote cylindrical coordinates about the x_3-axis such that $r = (x_1^2 + x_2^2)^{1/2}$ and such that $\varphi = 0$ (resp. $\varphi = \omega$) contains Γ_1 (resp. Γ_2). Denote by G an open subset of R^2 such that $G \times \gamma \subset \Omega$ and $\bar{G} \times \bar{\gamma}$ neither contains any corners of Ω nor meets any edges of Ω except for the edge containing γ. For each integer $m > 0$ and each integer $n \geq 0$, let

$$(4.2) \qquad z_{m,n}(r,\varphi) = r^{\frac{m\Pi}{\omega} + 2n} \sin(\frac{m\pi}{\omega}\,\varphi)$$

if $\frac{m\Pi}{\omega}$ is not an integer, and let

$$(4.3) \qquad z_{m,n}(r,\varphi) = r^{\frac{m\Pi}{\omega} + 2n} [\ln r \sin(\frac{m\Pi}{\omega}\,\varphi) + \cos(\frac{m\Pi}{\omega}\,\varphi)]$$

if $\frac{m\pi}{\omega}$ is an integer. The following result is contained in [16]:

<u>Lemma 4.1</u> Let f belong to $C^\infty(\bar{\Omega})$. Then for any positive integer k there exists[1] $h \in C^\infty(\bar{\gamma}, H^{k+2}(G))$ and $c_{m,n} \in C^\infty(\bar{\gamma})$ such that

$$(4.4) \qquad u = \sum_{0 < \frac{m\pi}{\omega} + 2n < k+1} c_{m,n}\, z_{m,n} + h$$

in $\overline{G \times \gamma}$.

<u>Remark</u> If one were only to assume that $f \in H^k(\Omega)$, Lemma 4.1 still holds except that in this case, one may only conclude that[2] $h \in L_2(\gamma, H^{k+2}(G))$ and $c_{m,n} \in H^{k+1-\frac{m\pi}{\omega}-2n}(\gamma)$ (see [16] and the references contained therein). The assumption that $f \in C^\infty(\bar{\Omega})$, which usually holds in practice, has been made in order to help simplify the presentation.

Together with Lemma 4.1, the following enables one to apply the approximation results of Section 2 to the solution u of (4.1) in a neighborhood of the edge γ.

<u>Lemma 4.2</u> Let $\xi_1, \xi_2 \in C^\infty([0, \omega])$, let $\eta \in C^\infty(\bar{\Omega})$ be such that supp $\eta \subset \overline{G \times \gamma}$, and let $c \in C^\infty(\bar{\gamma})$. Let v be a function on Ω such that, in $G \times \gamma$,

---

1 $C^\infty(X;Y)$ is the space of all infinitely differentiable functions on X whose values lie in Y.

2 $L_2(X;Y)$ is the space of all functions h on X with values in Y such that $\int_X \|h(x)\|_Y^2\, dx < \infty$.

$$v(r,\varphi,x_3) = c(x_3)r^{\mu}\,(\ln r\;\xi_1(\varphi) + \xi_2(\varphi))$$

for some non-negative real number μ. Then $v\eta \in Z_0^{2\mu+2-\epsilon}(\Omega;\Delta,D)$ for arbitrarily small $\epsilon > 0$.

<u>Pf</u>: Since v is smooth in the variable x_3, the result follows from Lemma 3.2.

Consider next the behavior of the solution u of (4.1) in a neighborhood of a corner of Ω, which may be taken to be the origin without loss of generality. Let γ denote an edge of Ω which has the origin as an endpoint; again assume that γ is contained in the positive x_3-axis and that ω is the measure of the interior dihedral angle of Ω at γ. Let K denote the intersection of Ω and a solid circular cone whose vertex is the origin, whose axis contains γ, and whose half-angle is sufficiently small that K does not meet any of the edges of Ω which have the origin as an endpoint except for the edge containing γ. Finally, let K_0 denote the intersection of K and a small sphere of radius ρ_0 about the origin. The present goal is to understand the behavior of u in K_0.

To this end, let G denote the section of the unit sphere intersected by the infinitely extended cone coinciding with Ω in a neighborhood of the origin, and let G_γ denote the intersection of K and the unit sphere. Denote by $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the eigenvalues of the Laplace-Beltrami operator on G with homogeneous Dirichlet boundary data on ∂G. Let (ρ,θ,φ) be the usual spherical coordinate system in R^3. Applying the previous two-dimensional results to the Laplace-Beltrami operator on G, let $\tilde{z}_{m,n}(\theta,\varphi)$ denote the singular functions analogous to (4.2), (4.3) where the singularity occurs at the corner $\theta=0$ of G. The following is proved in [16]:

Lemma 4.3 Let f belong to $C^{\infty}(\bar{\Omega})$ and let u be the solution of (4.1). Then for any integers k and k' satisfying $0 \le k' \le k$, there exist numbers $c_{m,n,i}$, d_i and functions $f_{m,n}$, g_i, h such that, in K_0,

$$u(\rho,\theta,\varphi) = \sum_{\frac{m\pi}{\omega}+2n<k+1} \Big[\sum_{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}<k} c_{m,n,i}\, \rho^{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}} \tag{4.5}$$

$$+ f_{m,n}(\rho)\Big] \tilde{Z}_{m,n}(\theta,\varphi) + \sum_{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}<k} d_i\, \rho^{-\frac{1}{2}+\sqrt{\lambda_i+\frac{1}{4}}} g_i(\theta,\varphi) + h(\theta,\varphi)$$

where $g_i \in H^{k+2}(G_\gamma)$ and for arbitrarily small $\varepsilon > 0$,

$$\rho^{-k+j-\frac{1}{2}+\varepsilon} \frac{\partial^j}{\partial\rho^j} f_{m,n} \in L_2(0,\infty) \qquad , \quad 0 \le j \le k' \quad ,$$

$$\rho^{-k+j-\frac{1}{2}+\varepsilon} \frac{\partial^j}{\partial\rho^j} h \in L_2\left((0,\infty);\, H^{k-k'+2}(G_\gamma)\right), \quad 0 \le j \le k' \quad .$$

The following lemma, together with Lemma 4.3, will allow the application of the approximation results of Section 2 to the solution u of (4.1) in K_0.

Lemma 4.4 Let $\eta_1 \in C^{\infty}(\bar{G})$ be such that supp $\eta_1 \subset \bar{G}_\gamma$ and let $\eta_2 \in C^{\infty}([0,\infty))$ be such that supp $\eta_2 \subset \{\rho:\ 0 \le \rho \le \rho_0\}$.

(i) If v is a function on Ω such that, in K_0,

$$\rho^{-\mu_1+\beta_1} \sin^{-\mu_2+\beta_2-1}\theta \frac{\partial^{|\underline{\beta}|} v}{\partial\rho^{\beta_1}\partial\theta^{\beta_2}\partial\varphi^{\beta_3}} \in L_2((0,\infty);\, L_2(G)) \ , \tag{4.6}$$

$$0 \le |\underline{\beta}| = \beta_1 + \beta_2 + \beta_3 \le \nu \quad ,$$

for some positive real numbers μ_1, μ_2 and for some positive integer ν, then $v\eta_1\eta_2 \in Z_0^{2\min(\mu_1,\mu_2)+2-\varepsilon}(\Omega;\Delta,D)$ for arbitrarily small $\varepsilon > 0$.

(ii) If v is a function on Ω such that, in K_0,

$$\rho^{-\mu+\beta_1} \frac{\partial^{|\underline{\beta}|} v}{\partial\rho^{\beta_1} \partial\theta^{\beta_2} \partial\varphi^{\beta_3}} \in L_2((0,\infty); L_2(G)) \quad ,$$

$$0 \leq |\underline{\beta}| = \beta_1 + \beta_2 + \beta_3 \leq \nu \quad ,$$

for some positive real μ and some positive integer ν, then $v\eta_1\eta_2 \in Z_0^{2\mu+2-\varepsilon}$ $(\mathcal{L};\Delta,D)$ for arbitrarily small $\varepsilon > 0$.

Pf: Assume that the Cartesian coordinate system (x_1, x_2, x_3) has been transformed by an affine mapping in such a way that ρ now represents the distance to the point $(1,1,1)$ and that θ is the angle measured from the semi-axis $\{x: x_1 = x_2 = 1, x_3 < 1\}$. To prove (i), it suffices to show that a function v in I^3 satisfying (4.6) (with the new definition of (ρ,θ,φ) and corresponding redefinition of G_γ) belongs to $Z^{2\min(\mu_1,\mu_2)+2-\varepsilon}(I^3)$ for arbitrarily small $\varepsilon > 0$.

To this end, let $\chi \in C^\infty([0,\infty))$ be such that

$$\chi(t) = \begin{cases} 0 & \text{for } 0 \leq t < 1/2 \\ 1 & \text{for } 1 \leq t < \infty \end{cases}$$

and for each $\delta > 0$ let $\chi_\delta(t) = \chi(\frac{t}{\delta})$. Set $v_\delta =$ $v\,\chi_\delta(\rho)\,\chi_\delta(\rho\sin\theta)$. Then, for $i = 1, 2, 3$,

$$\left|\frac{\partial^k v_\delta}{\partial x_i^k}\right| \leq C \sum_{|\underline{\alpha}|=k} \left|\frac{\partial^{\alpha_1} v}{\partial x_i^{\alpha_1}} \frac{\partial^{\alpha_2}\chi_\delta(\rho)}{\partial x_i^{\alpha_2}} \frac{\partial^{\alpha_3}\chi_\delta(\rho\sin\theta)}{\partial x_i^{\alpha_3}}\right|$$

$$\leq C \sum_{\alpha_1=0}^{k} \left|\frac{\partial^{\alpha_1} v}{\partial x_i^{\alpha_1}}\right| \delta^{\alpha_1-k}$$

and

$$\int_{I^3} \left|\frac{\partial^k v_\delta}{\partial x_i^k}\right|^2 (1 - x_i^2)^k dx \leq C \sum_{\alpha_1=0}^{k} \delta^{2\alpha_1-2k} \int_{\delta/2<\rho<\delta} \left|\frac{\partial^{\alpha_1} v}{\partial x_i^{\alpha_1}}\right|^2 (1 - x_i)^k dx$$

$$\leq C \sum_{\alpha_1=0}^{k} \delta^{2\alpha_1-2k} \sum_{|\underline{\beta}|=\alpha_1} \int_{\frac{\delta}{2}<\rho<\delta} \left| \frac{\partial^{\alpha_1} v}{\partial\rho^{\beta_1}\partial\theta^{\beta_2}\partial\varphi^{\beta_3}} \right|^2 \rho^{-2\beta_2-2\beta_3+k+2} \sin\theta \, d\rho d\theta d\varphi$$

$$\leq C \sum_{\alpha_1=0}^{k} \delta^{2\alpha_1-2k} \sum_{|\underline{\beta}|=\alpha_1} \int_{\frac{\delta}{2}<\rho<\delta} \left| \frac{\partial^{\alpha_1} v}{\partial\rho^{\beta_1}\partial\theta^{\beta_2}\partial\varphi^{\beta_3}} \right|^2 \rho^{-2\mu_1+2\beta_1} \sin^{-2\mu_2+2\beta_2-2}\theta$$

$$\cdot\ \rho^{2\mu_1+k-2\alpha_1+2} \sin^{2\mu_2-2\beta_2+2}\theta \sin\theta \, d\rho d\theta d\varphi \tag{4.7}$$

$$\leq C \sum_{\alpha_1=0}^{k} \delta^{2\alpha_1-k} \sum_{|\underline{\beta}|=\alpha_1} \int_{\frac{\delta}{2}<\rho<\delta} \left| \frac{\partial^{\alpha_1} v}{\partial\rho^{\beta_1}\partial\theta^{\beta_2}\partial\varphi^{\beta_3}} \right|^2 \rho^{-2\mu_1+2\beta_1} \sin^{-2\mu_2+2\beta_2-2}\theta$$

$$\cdot\ (\rho\sin\theta)^{-2\alpha_1} \rho^{2\mu_1+2} \sin^{2\mu_2+2}\theta \sin\theta \, d\rho d\theta d\varphi$$

$$\leq C \sum_{\alpha_1=0}^{k} \delta^{2\alpha_1-k-2\alpha_1+2\min(\mu_1,\mu_2)+2}$$

$$\leq C\, \delta^{2\min(\mu_1,\mu_2)+2} \quad .$$

Moreover,

$$\int_{I^3} |v - v_\delta|^2 \, dx = \int_{I^3} |v\,(1 - \chi_\delta(\rho)\chi_\delta(\rho\sin\theta))|^2 \, dx$$

$$\leq C \int_{\rho\sin\theta<\delta} |v|^2 \, dx$$

$$\leq C \int_{\rho\sin\theta<\delta} |v|^2 \rho^{-2\mu_1} \sin^{-2\mu_2-2}\theta \, \rho^{2\mu_1+2} \sin^{2\mu_2+2}\theta \sin\theta \, d\rho d\theta d\varphi \tag{4.8}$$

$$\leq C \int_{\rho\sin\theta<\delta} |v|^2 \rho^{-2\mu_1} \sin^{-2\mu_2+2}\theta \, (\rho\sin\theta)^{2\min(\mu_1,\mu_2)+2} \sin\theta \, d\rho d\theta d\varphi$$

$$\leq C\, \delta^{2\min(\mu_1,\mu_2)+2} \quad .$$

For $t > 0$, define

$$K(v,t) = \inf_{v=v_1+v_2} \left(\|v_1\|_{L_2(I^3)} + t\, \|v_2\|_{Z^k(I^3)} \right) \quad .$$

For $0 < t < 1$ let $v_1 = v - v_\delta$ and $v_2 = v_\delta$. Then, by (4.7) and (4.8),

$$K(v, t) \leq C(\delta^{\min(\mu_1,\mu_2)+1} + t\,\delta^{-k/2 + \min(\mu_1,\mu_2)+1}) .$$

Choosing $\delta = t^{2/k}$, it follows that

$$K(v, t) \leq C\, t^{\frac{2\min(\mu_1,\mu_2)+2}{k}} , \qquad 0 < t < 1 .$$

For $1 \leq t < \infty$, take $v_1 = v$ and $v_2 = 0$ to obtain

$$K(v, t) \leq C\, \|v\|_{L_2(I^3)} , \qquad 1 \leq t < \infty .$$

Hence,

$$\int_0^\infty (t^{-\theta} K(v, t))^2 \frac{dt}{t} \leq C\int_0^1 t^{-2\theta + \frac{4\min(\mu_1,\mu_2)+4}{k} - 1}\, dt$$

$$+ C\, \|v\|_{L_2(I^3)} \int_1^\infty t^{-2\theta - 1} dt < \infty$$

provided that $0 < \theta < \frac{2\min(\mu_1,\mu_2)+2}{k}$. By the definition of real interpolation via the K-method together with [11, Theorem 2.1], it follows that for arbitrarily small $\varepsilon > 0$,

$$v \in (L_2(I^3), Z^k(I^3))_{\frac{2\min(\mu_1,\mu_2)+2-\varepsilon}{k}, 2}$$

$$= Z^{2\min(\mu_1,\mu_2)+2-\varepsilon}(I^3)$$

which completes the proof of (i).

The proof of (ii) is essentially the same as that of part (i) except that one instead sets $v_\delta = v\chi_\delta(\rho)$.

Having established the regularity of the solution u of (4.1) near the edges and corners of Ω, the piecewise polynomial approximability of u in $H^1(\Omega)$ can now be determined. Let σ_ℓ, $1 \leq \ell \leq L$, denote the corners of Ω and let $0 < \lambda_{1,\ell} \leq \lambda_{2,\ell} \leq \cdots$ denote the eigenvalues of the Laplace-Beltrami operator on G_ℓ with homogeneous Dirichlet boundary conditions on ∂G_ℓ where G_ℓ is the portion of the unit sphere subtended by the infinite cone which coincides with Ω in a neighborhood of σ_ℓ. Let γ_j, $1 \leq j \leq J$, denote the edges of Ω and let ω_j denote the measure of the interior dihedral angle of Ω at γ_j. The following is the main result of this section.

<u>Theorem 4.1</u> Let f belong to $C^\infty(\bar{\Omega})$. If u is the solution of (4.1) then for each non-negative integer p there exists $\psi_p \in P_p^0(\Omega;\Delta,D)$ such that

$$\|u - \psi_p\|_{H^1(\Omega)} \leq C(u, \varepsilon)p^{-2\mu + \varepsilon}$$

for arbitrarily small $\varepsilon > 0$ where

$$(4.9) \qquad \mu = \min_{\substack{1\leq\ell\leq L \\ 1\leq j\leq J}} \left\{ \frac{\pi}{\omega_j}, -\frac{1}{2} + \sqrt{\lambda_{1,\ell} + \frac{1}{4}} \right\}$$

and $C(u, \varepsilon)$ is independent of p.

<u>Pf</u>: By employing an appropriate partition of unity, the approximability of u in Ω may be determined by separately considering the approximability of u near the edges of Ω, near the corners of Ω, and in the interior of Ω. Since u is infinitely differentiable in the interior of Ω, Theorem 2.2 may be applied with any value of s yielding an arbitrarily high degree of approximation there. Near an edge γ_j, Lemma 4.2 and Theorem 2.2 (with an arbitrarily large value of s) may be applied to the terms $z_{m,n}$ and h of

(4.4) respectively; this together with Theorem 2.1 yields an approximation degree of $\frac{2\pi}{\omega_j} - \epsilon$ near γ_j. Near a corner σ_ℓ and an edge γ_j with σ_ℓ as an endpoint, one applies Lemma 4.4 to each term of (4.5) with $k' = [\frac{k}{2}]$, k arbitrarily large, and μ_1, μ_2, μ, ν chosen as follows:

1. For the term $\rho^{-\frac{1}{2} + \sqrt{\lambda_{i,\ell} + \frac{1}{4}}}\tilde{z}_{m,n}$, take $\mu_1 = -\frac{1}{2} + \sqrt{\lambda_{i,\ell} + \frac{1}{4}}$, $\mu_2 = \frac{m\pi}{\omega_j} + 2n$, and ν arbitrary in (i);
2. For the term $f_{m,n}\tilde{z}_{m,n}$, take $\mu_1 = k$, $\mu_2 = \frac{m\pi}{\omega_j} + 2n$, and $\nu = k'$ in (i);
3. For the term $\rho^{-\frac{1}{2} + \sqrt{\lambda_{i,\ell} + \frac{1}{4}}} g_i$, take $\mu = -\frac{1}{2} + \sqrt{\lambda_{i,\ell} + \frac{1}{4}}$ and $\nu = k + 2$ in (ii);
4. For the term h, take $\mu = k$ and $\nu = k'$ in (ii).

Theorem 2.1 then yields an approximation degree of $2 \cdot \min\left\{\frac{\pi}{\omega_j}, -\frac{1}{2} + \sqrt{\lambda_{1,\ell} + \frac{1}{4}}\right\} - \epsilon$ for u near σ_ℓ and γ_j. The proof is completed by adding together the various approximating piecewise polynomials obtained in this manner and applying the triangle inequality.

In the two-dimensional result Theorem 3.1, the degree of approximation is dependent solely upon the measures ω_j of the interior angles at the corners of the domain and upon the boundary conditions; since these are always known, the approximability of the solution of (3.1) is readily computed in all cases. While the effect of singularities along the edges of a polygonal domain is similarly expressed in terms of the dihedral angle measures ω_j in Theorem 4.1, the presence of the Laplace-Beltrami eigenvalues in (4.9) makes the determination of the approximation degree more difficult in the three-dimensional case since the relationship between these eigenvalues and known quantities in the problem is not clear. The determination of the smallest (or fundamental) eigenvalue λ_1 of the Laplace-Beltrami operator on various domains has been considered by a number of authors (see e.g. [8],

[20],[27]). For some configurations, λ_1 can be obtained analytically, however, in general one must resort to numerical techniques. Figure 4.1 below depicts a solid Ω containing a notch and rectangular crack. Assuming that (4.1) is posed on Ω, Table 4.1 lists the values of λ_1 corresponding to each of the corners of Ω together with the method used to compute λ_1 and the quantity $-\frac{1}{2}+\sqrt{\lambda_1+\frac{1}{4}}$. Considering Table 4.1 together with (4.9), it is interesting to note that the corners of type A, B, or C, the degree of approximation is not affected by the "corner singularities" since these are weaker than the singularities along the edges converging at such corners. On the other hand, for corners of type D or E, the corner singularities are dominant.

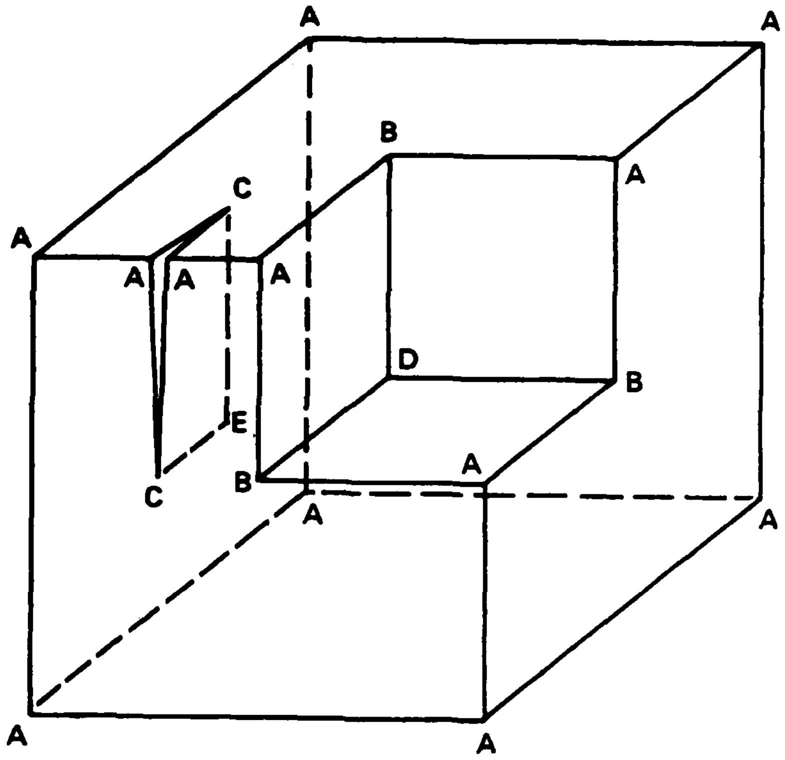


Figure 4.1: Solid with notch and rectangular crack.

Corner	λ_1	Method of Computing λ_1	$-\frac{1}{2}+\sqrt{\lambda_1+\frac{1}{4}}$
A	12.000	Analytical [27]	3.000
B	1.667	Analytical [8]	0.884
C	1.500	Analytical [8]	0.823
D	0.453	Numerical [8]	0.338
E	0.396	Numerical [27]	0.304

Table 4.1

5. Numerical Results

The relevance of any asymptotic error analysis such as the one presented here depends of course on the ability of the underlying method to achieve the predicted asymptotic order within practical computational limits. Since these issues are difficult to resolve analytically, one must rely on numerical experiments to determine whether it is possible to see the predicted asymptotic behavior of the error without using an unreasonable number of degrees of freedom. Such experiments also provide useful information about the level of accuracy that can be expected from a given amount of computational effort. In this section, the p-version of the finite element method is applied to two model problems from two-dimensional linear elasticity using the research program COMET-X developed at the Center for Computational Mechanics at Washington University in St. Louis [7]. While the computations do agree with the asymptotic orders predicted by the above approximation theory, the main conclusion to be drawn from these results is that for problems of practical interest, the p-version does in fact enter the asymptotic range for low values of p. Moreover, it is seen that the order of convergence is completely determined by point singularities in the solutions, and that, unlike the h-version, the point at which the p-version enters the asymptotic range is virtually unaffected by the Poisson ratio.

In order to link the following computations with the approximation theory of the previous sections, a number of preliminary remarks must be made.

(i) The approximation results of [11] were developed only for scalar problems. However, there are no difficulties in generalizing these results to the approximation of vector-valued functions (such as the two-dimensional displacement fields treated below) by vectors of piecewise polynomials in the

Sobolev product spaces $\mathbb{H}^s = \Pi\, H^s$.

(ii) Along with a table of displacements and stresses, COMET-X calculates the strain energy

$$J(\underline{u}_p) = 1/2 \int_\Omega [D\underline{u}_p]^T B [D\underline{u}_p] d\Omega$$

of the computed solution $\underline{u}_p$ where D is a differential operator and B is an elasticity matrix which depends on the elasticity modulus E, the Poisson ratio ν, and the type of two-dimensional problem being considered, i.e. either plane strain or plane stress (for a more detailed description of the matrices D and B, the reader is referred to [28]). In terms of J, one defines the energy norm

$$\|\cdot\| = J(\cdot)^{1/2}$$

which is equivalent to the usual $\mathbb{H}^1 = H^1 \times H^1$ Sobolev norm on the space of admissible displacements with rigid body motions factored out. Since the error in the energy norm corresponds to a least squares error in the stresses, it is generally desired in engineering applications to achieve 5 to 10 percent relative error in this norm. This, of course, corresponds to a .25 to 1 percent relative error in energy.

(iii) In two dimensions, $(p+1)(p+2)/2$ degrees of freedom are required to uniquely specify a polynomial of degree p. Since the p-version assumes a fixed mesh, it follows that the dimension N of the finite element subspaces grows asymptotically like p^2 as $p \to \infty$. This implies that if the theory of the previous sections predicts an error estimate of the form

$$\| \underline{u} - \underline{u}_p \|_{H^1} \leq Cp^{-\alpha}, \ \alpha > 0,$$

where $\underline{u}$ is the exact solution, then the corresponding estimate of the error in energy with respect to the number of degrees of freedom is given by

$$J(\underline{u} - \underline{u}_p) \leq CN(p)^{-\alpha}.$$

Due to a well-known orthogonality property of the error $\underline{u} - \underline{u}_p$, this is the same as

$$(5.1) \qquad | J(\underline{u}) - J(\underline{u}_p) | \leq CN(p)^{-\alpha} .$$

Consider an edge-cracked square panel as depicted in Figure 5.1. In order to have an exact energy with which to compare energies computed by COMET-X, let it be assumed that the tractions σ_i, $i = 1, \ldots, 5$ are such that if $\underline{u}$ is the restriction to the panel of the exact solution (as constructed e.g. in [21,sec. 120]) for an infinite cracked panel, then $\underline{u}$ solves the finite problem of Figure 5.1. Near the crack tip, $\underline{u}$ is of the form $r^{1/2}\underline{\Phi}(\theta)$ where (r,θ) are polar coordinates at the tip of the crack and $\underline{\Phi}$ is a smooth vector function. Using the formula for $\underline{u}$, one computes $J(\underline{u}) = 42.16$.

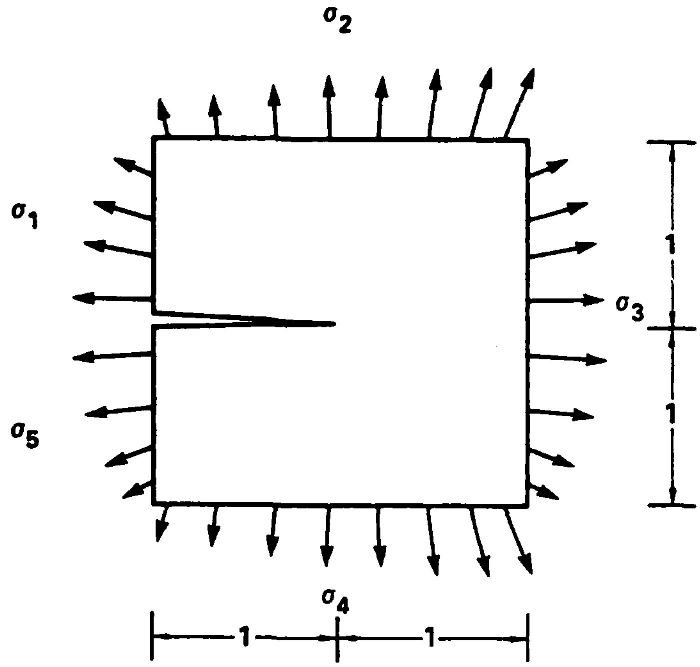
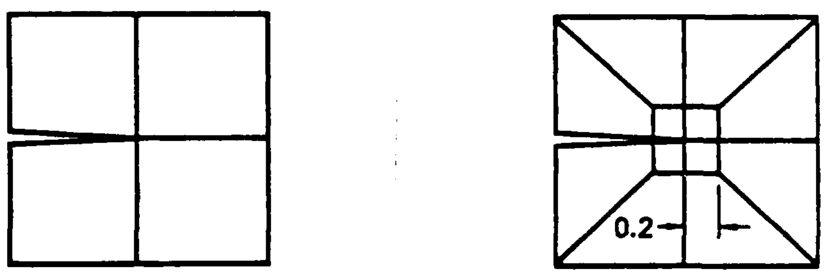


Figure 5.1: An edge-cracked panel. Elasticity modulus E = 1.0, Poisson ratio ν = 0.3, Traction σ as specified in the text, Plane strain.



Mesh # 1 Mesh # 2

Figure 5.2: Two meshes for the edge-cracked panel of Figure 5.1.

Applying the p-version to this problem, one subdivides the panel in some convenient manner. Two possible meshes are shown in Figure 5.2. It should be noted here that although the approximation theory presented in [11] has assumed triangular subdivisions this is not essential, and the results presented are equally valid for quadrilateral subdivisions such as in Figure 5.2. By the results of section 3, one obtains that, neglecting ε, for each polynomial degree p, the p-version finite element solution $\underline{u}_p$ satisfies

$$\| \underline{u} - \underline{u}_p \|_{H^1} \leq Cp^{-1}$$

where C is independent of p. Hence the error in energy is expected to decrease linearly with respect to the number of degrees of freedom.

Figure 5.3 shows the COMET-X results for this problem. One observes that the predicted linear convergence is achieved and that the asymptotic range is entered for p as low as 4 for either mesh #1 or mesh #2. This entry into the asymptotic range therefore occurs when the relative energy norm error is about 27 percent and 14 percent for mesh #1 and mesh #2, respectively. The superiority of mesh #2 over mesh #1 is clearly due to the fact that, given a fixed number of degrees of freedom, mesh #2 allows more degrees of freedom to participate in the resolution of the singularity at the crack tip than does mesh #1. Note that for mesh #2 a relative error in energy of less than one percent, i.e. a relative energy norm error of less than 10 percent, is achieved with $p = 6$, which corresponds to 482 degrees of freedom. Furthermore, if the error curve for mesh #2 is linearly extended, one finds that a 5 percent relative energy norm error is achievable by using less than 2,000 degrees of freedom. A problem of this size could be handled by a large mainframe.

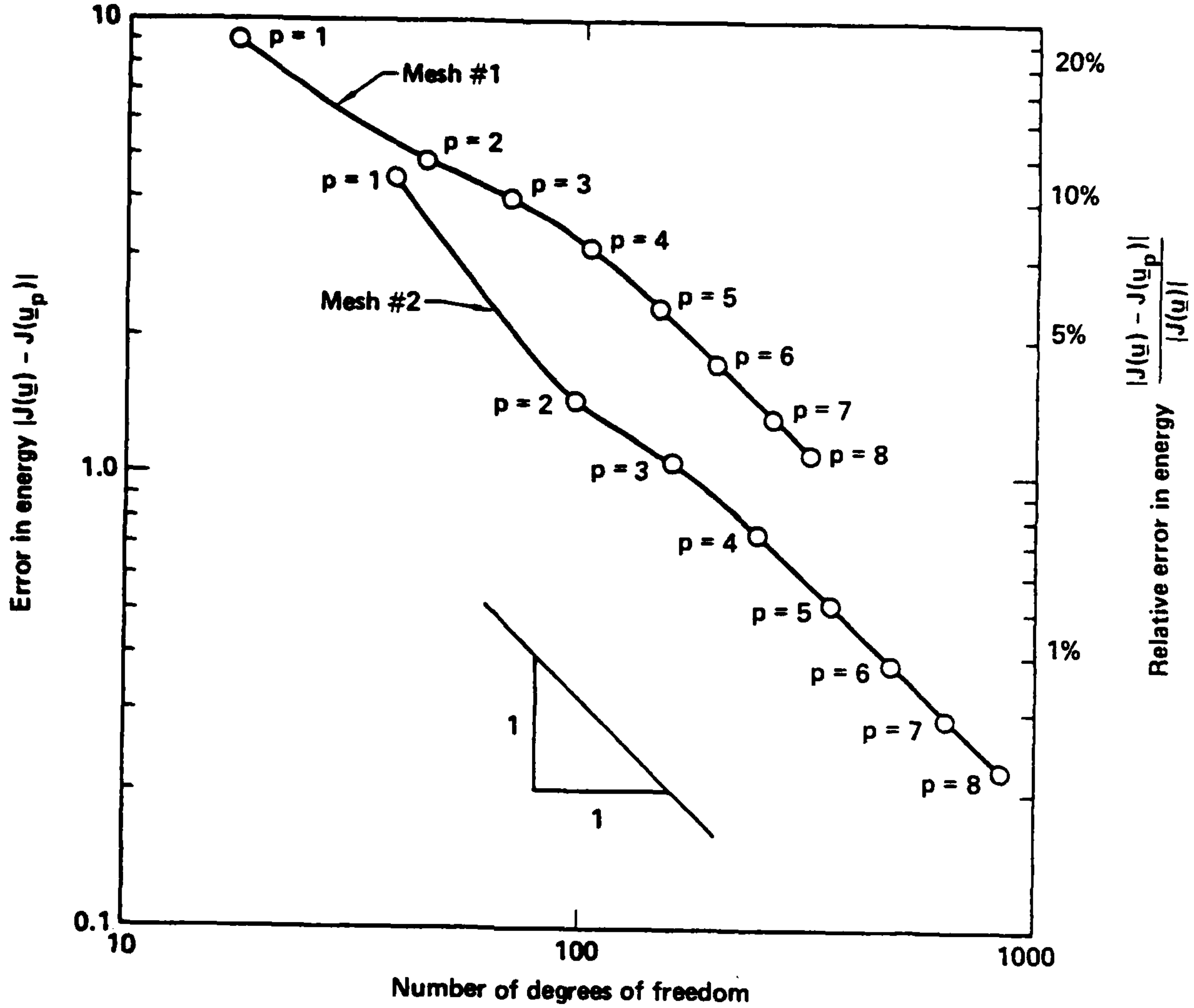


Figure 5.3: COMET-X results for
the edge-cracked panel of Figure 5.1.

Consider next the edge-cracked square panel in Figure 5.4 where the lower side of the crack is held fixed and part of the upper edge of the panel is subjected to a uniform traction as shown. Although there does not exist an analytical solution to this problem, it can be rigorously demonstrated that near the crack tip, the displacement field $\underline{u}$ is of the form $r^{\alpha}\underline{\Phi}(\theta)$ where (r,θ) are polar coordinates based at the crack tip, $\underline{\Phi}$ is a smooth vector function, and $\alpha \approx 1/4$. Milder singularities of a similar form exist at the other corners of the panel. By the approximation theory of section 3, it holds that, neglecting ϵ, for each polynomial degree p, the p-version finite element solution $\underline{u}_p$ satisfies

$$\| \underline{u} - \underline{u}_p \|_{H^1} \leq C p^{-1/2}$$

where C is independent of p. A convergence order of $1/2$ with respect to the number of degrees of freedom is therefore expected.

Since the exact energy is unknown for this problem, some other means were required to obtain a reasonable estimate of this value. For this purpose, the self-adaptive h-version code FEARS (Finite Element Adaptive Research Solver) developed at the University of Maryland [5] was used. Since FEARS provides an aposteriori estimate of the error which may be employed to obtain a prediction of the exact energy, this procedure was used to obtain the values $J(\underline{u}) = 1.57$ for $\nu = .3$ and $J(\underline{u}) = 1.23$ for $\nu = .49$.

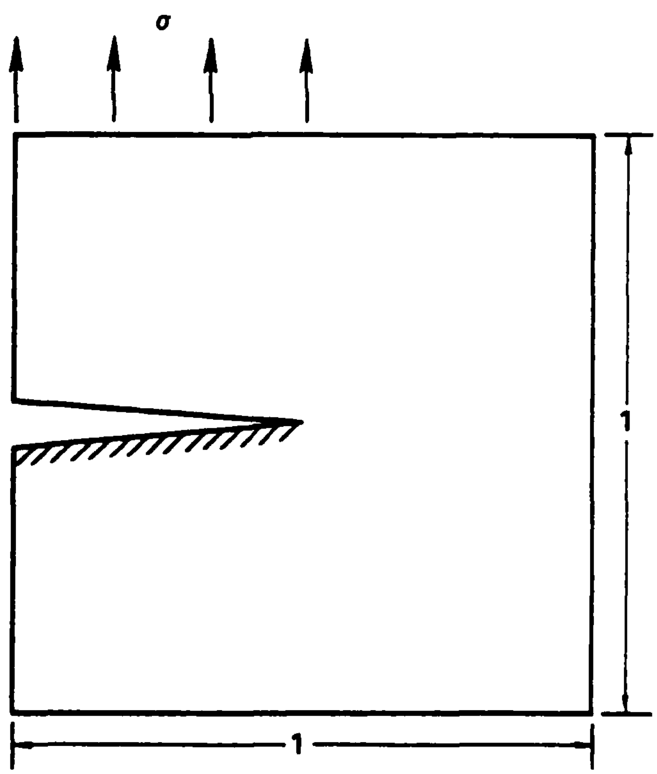


Figure 5.4: An edge-cracked panel with constrained crack. Elasticity modulus $E = 1.0$, Poisson ratio $\nu = 0.3$ or 0.49, Traction $\sigma = 1.$, Plane strain.

Figure 5.5 contains the COMET-X and FEARS results for this problem using Poisson ratios of $\nu = .3$ and $\nu = .49$. The COMET-X triangulation of the panel is shown in the inset. Since FEARS adaptively selected a number of meshes to obtain the results of Figure 5.5, these are not shown.

One observes that the COMET-X results are quite similar for $\nu = .3$ and $\nu = .49$. As for FEARS, the results for $\nu = .49$ are somewhat worse than those for $\nu = .3$, although the fact that an asymptotic rate is even observed is rather significant since non-adaptive h-version codes are known to perform poorly for values of ν near the incompressibility limit 1/2. The invariance of the p-version with respect to the Poisson ratio ν has been previously observed by other investigators and is rigorously analyzed in [26].

The convergence order for COMET-X is 1/2, as predicted, whereas the convergence order for FEARS is 1, which is expected due to the fact that the adaptive procedure eliminates the effect of the singularity on the order of convergence leaving only a dependence on the polynomial degree (=1). For $\nu = .3$, the COMET-X error is less than that of FEARS up to about 350 degrees of freedom, and for $\nu = .49$ the COMET-X error is less than the FEARS error up to about 2500 degrees of freedom. These cross-over points correspond to a relative energy error of about 9.5 percent and 3.7 percent, respectively. Thus, in very low accuracy ranges, the COMET-X results are somewhat better especially for the Poisson ratio $\nu = .49$. On the other hand, in the higher accuracy ranges FEARS achieves a lower error than COMET-X. In fact, by linearly extending the error curves of Figure 5.4, one easily checks that in order to obtain results which have a relative error of 10 percent in the energy norm, if $\nu = .3$ then COMET-X would require about 35,000 degrees of freedom as opposed to 3,000 for FEARS. If $\nu = .49$ then COMET-X would require about 25,000 degrees of freedom to about

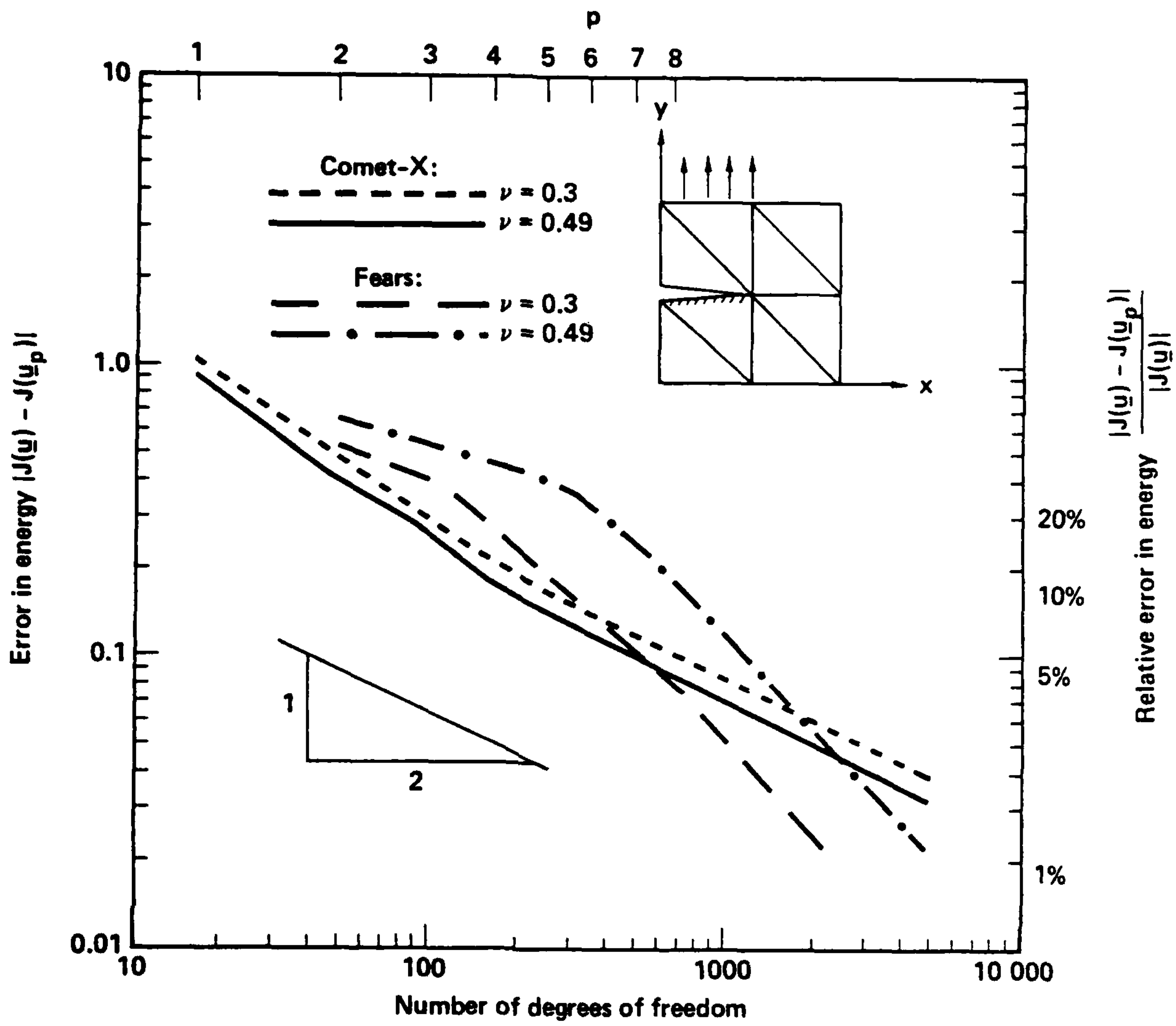


Figure 5.5: COMET-X and FEARS results
for the edge-cracked panel of Figure 5.4.

7,000 for FEARS. Both of the latter numbers of degrees of freedom for COMET-X must be regarded as being outside practical computational limits.

In summary, although the above numerical results are in no way exhaustive, they do suggest the following:

(i) The order of convergence of the p-version agrees with that predicted by the approximation theory. Moreover, this order of convergence is completely controlled by a principal singularity occurring at a point(s) on the boundary.

(ii) The asymptotic range is achieved for low values of p. Furthermore, the asymptotic range has generally been entered by the time the relative error in the energy norm has reached 10 to 30 percent.

(iii) The p-version is not sensitive to changes in the Poisson ratio ν, even when ν is close to the incompressibility limit.

(iv) If the dominant singularity is of the form r^{α} with $\alpha \geq 1/2$, then it is possible to achieve a relative error in the energy norm of 5 percent without exceeding practical computational limits. If, however, $\alpha < 1/2$, say e.g. $\alpha = 1/4$ as in the second example, then it may not be possible to obtain a relative energy norm error of less than 20 percent without using an excessive number of degrees of freedom. For problems with such strong singularities, it seems clear that an adaptive p-version or a combination of h- and p-versions [3] will be necessary to obtain the level of accuracy required for many problems.

Of course, the above considerations have avoided other aspects which figure significantly in the actual cost of finite element analysis. These include the sparsity of the resulting matrices, the complexity of the input data, issues of data structure and data management, etc. Questions such as these are discussed in [6] and [22-24].

6. Acknowledgement

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